PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
ESCUELA DE INGENIERIA
DEPARTAMENTO DE CIENCIA DE LA COMPUTACION

Complexity Theory, Semester I 2017 - IIC3242
Homework 5
Deadline: Thursday, May 18th, 2017

1 The first two levels of PH [4 points]

Let φ be a propositional formula *in conjunctive normal form*, and σ a valuation. We say that σ is **optimal for** φ, if the number of clauses of φ satisfied by σ is k, and there is no valuation σ' that satisfies $k+1$ or more clauses of φ. Note that if the entire formula φ is satisfiable, then any assignment satisfying it is optimal. The more interesting formulas are the ones not satisfiable.
As an example, take the formula $\varphi = (x \vee y \vee z) \wedge (\neg x \vee \neg y) \wedge (x \vee \neg y) \wedge (\neg x \vee y) \wedge (x \vee y)$. Here we have that a valuation σ with $\sigma(z) = 1$ is optimal, as it always makes 4 clauses true (irrespective of the values assigned to x and y). On the other hand, the valuation $\sigma(x) = \sigma(y) = \sigma(z) = 0$ makes only 3 clauses true.
Consider the following problem:

$$MAX - SAT = \{(\varphi, k) : \varphi \text{ is a formula, and every optimal assignment for } \varphi \text{ satisfies precisely } k \text{ clauses}\}.$$

a) Show that $MAX - SAT \in \Sigma_2^P$. [1 point]

b) Show that $MAX - SAT \in$ NP if and only if NP = co-NP. [3 points]

2 Third level of PH [3 points]

Let P be a set of propositional variables. Denote by $F(P)$ the set of all propositional formulas that use the variables in P. For a finite set $\Sigma \subseteq F(P)$ and a formula $\varphi \in F(P)$ we define the following set of sets:

$$W(\Sigma, \varphi) = \{\Sigma' \subseteq \Sigma : \Sigma' \not\models \neg\varphi \text{ and for all } \Gamma \text{ such that } \Sigma' \subsetneq \Gamma \subseteq \Sigma \text{ it holds that } \Gamma \models \neg\varphi\}.$$

Intuitively, the set $W(\Sigma, \varphi)$ contains all the maximal subsets of Σ that are consistent with φ. Let us define the following problem called $KNOWLEDGE - REVISION$:

$$KNOWLEDGE - REVISION = \{(\Sigma, \varphi, \psi) : \bigcap W(\Sigma, \varphi) \models \psi\},$$

where $\bigcap W(\Sigma, \varphi) = \bigcap_{\Sigma' \in W(\Sigma, \varphi)} \Sigma'$. We assume Σ to be finite as usual.

Show that $KNOWLEDGE - REVISION$ belongs to the class Δ_3^P.